A GENERAL SEARCH GAME*

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ABSTRACT

The minimax solution is found for a game in which player I chooses a real number and player II seeks it by choosing a trajectory represented by a positive function.

1. Introduction and presentation of main results

The Linear Search Game solved by Beck and Newman [1] and the extensions treated by the author [3] are special cases of the following game:

Player I chooses a number $-\infty < t < \infty$ and player II chooses a positive function $r(t)$ which will be called: "The search trajectory". The loss of player II is

$$M(r(t),t) = \frac{\int_{-\infty}^{\infty} r(t+\theta)dA(\theta)}{r(t)} \tag{1}$$

where $A(\theta)$ is the distribution function of a (fixed) positive measure. We shall show that the exponential function

$$r(t) = C\mathrm{e}^{bt} \tag{2}$$

is a (pure) minimax search trajectory of player II. We shall also find conditions under which (2) is the unique solution, up to a multiplicative constant.

Our main results are the following:

THEOREM 1. *Let $A(\theta)$, $-\infty < \theta < \infty$, be any non-decreasing function. Let $r(t)$, $-\infty < t < \infty$, be a positive function which is integrable on every finite interval. If*

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(3) $$s(t) = \int_{-\infty}^{\infty} r(t+\theta)dA(\theta)$$

is defined for all real t, then

(4) $$\limsup_{t\to-\infty} \frac{s(t)}{r(t)} \geqq \inf_{-\infty \leqq b \leqq \infty} \int_{-\infty}^{\infty} e^{b\theta} dA(\theta)$$

and

(4′) $$\limsup_{t\to+\infty} \frac{s(t)}{r(t)} \geqq \inf_{-\infty \leqq b \leqq \infty} \int_{-\infty}^{\infty} e^{b\theta} dA(\theta).$$

THEOREM 2. *Let $A(\theta)$ and $r(t)$ be defined as in Theorem 1. In addition assume that $A(\theta)$ is not concentrated at $\theta = 0$. Let $g(b)$ be the Bilateral Laplace Transform of $A(\theta)$ defined as follows:*

(5) $$g(b) = \int_{-\infty}^{\infty} e^{b\theta} dA(\theta).$$

Assume that $g(b)$ attains its minimum at a point $-\infty < \bar{b} < \infty$ so that

(6) $$\int_{-\infty}^{\infty} \theta e^{\bar{b}\theta} dA(\theta) = 0.$$

If

(7) $$\sup_{-\infty < t < \infty} \frac{\int_{-\infty}^{\infty} r(t+\theta)dA(\theta)}{r(t)} \leqq g(\bar{b})$$

then

(a) *If $A(\theta)$ is not arithmetic*, $r(t) = Ce^{\bar{b}t}$* a.s. *where C is any constant.*

(b) *If $A(\theta)$ is arithmetic with span λ, $r(t) = C(t)e^{\bar{b}t}$ where $C(t)$ is a periodic function having period λ.*

Both Theorems 1 and 2 hold for the discrete case (i.e. where $A(\theta)$ and $r(t)$ are replaced by positive sequences). A detailed formulation of the discrete version will be given in Chapter 4.

Chapter 5 will contain some examples which will clarify the use of the theorems. A vivid illustration of the application of the theorems for several search games is presented in [3].

* We use arithmetic in the sense used in Feller [2], namely: A distribution A is arithmetic if it is concentrated on a set of points of the form 0, $\pm\lambda$, $\pm 2\lambda$, ... The largest λ with this property is called the span of A.

2. Three fundamental Lemmas

LEMMA 1. *Let $P(\theta)$, $-\infty < \theta < \infty$, be the distribution function of a probability measure which is not concentrated at $\theta = 0$, satisfying:*

$$\int_{-\infty}^{\infty} \theta dP(\theta) = 0 \tag{8}$$

and let $w(t)$, $-\infty < t < \infty$, be a positive function which is integrable in each finite segment. If

$$l(t) = \int_{\infty}^{\infty} w(t+\theta) dP(\theta) \tag{9}$$

is defined for each t, then

$$\limsup_{t \to +\infty} \frac{l(t)}{w(t)} \geqq 1 \tag{10}$$

and

$$\limsup_{t \to -\infty} \frac{l(t)}{w(t)} \geqq 1. \tag{10'}$$

PROOF. First we note that on grounds of symmetry, it is enough to establish (10). We may assume that

(11) $P(\theta)$ has no positive probability mass at the origin (otherwise we can consider the conditional probability measure $P(\theta / \theta \neq 0)$).

The proof is given in three stages:

(a) First we prove (10) under two additional conditions viz:

(12) $w(t)$ is continuous, and

(13) $P(\theta)$ is supported by a finite segment $[-L, L]$ $(L > 0)$.

In case either

$$\liminf_{t \to +\infty} \frac{w(t)}{t} = \infty \tag{14}$$

or

$$\liminf_{t \to +\infty} w(t) = 0 \tag{15}$$

holds, we define $w^*(t)$ to be the maximal convex function satisfying

$$w^*(t) \leqq w(t), \text{ for all } t \geqq 0. \tag{16}$$

Condition (14) (or (15)) implies that for each t there exists a $t_1 > t$ satisfying

$$w^*(t_1) = w(t_1). \tag{17}$$

It follows from (8) that

$$l(t_1) \geqq w^*(t_1) = w(t_1) \tag{18}$$

so that (10) holds. Therefore, we may assume that

$$\liminf_{t\to\infty} \frac{w(t)}{t} = C < \infty \tag{19}$$

and that

$$\liminf_{t\to\infty} w(t) > 0. \tag{20}$$

If (10) were false, there would exist a T such that for all $t \geqq T$

$$l(t) < dw(t), \qquad 0 < d < 1. \tag{21}$$

It follows from (21) that for each $t \geqq T$, there exists an a_t satisfying

$$w(a_t) < dw(t) \tag{22}$$

where

$$|a_t - t| \leqq \mathrm{L} \ (\mathrm{L} \text{ being defined by (13)}). \tag{23}$$

It follows from (19) that for each $t \geqq T$, there exists a $t_1 > t$ satisfying

$$w(t_1) < (C + \varepsilon)t_1. \tag{24}$$

Let

$$n = [(t_1 - T)/L]. \tag{25}$$

By applying (22) n times, we find a $t_2 \geqq 0$ satisfying

$$w(t_2) < d^n w(t_1) < d^{(t_1 - T - L)/L}(C + \varepsilon)t_1. \tag{26}$$

It follows from (26) that:

$$\inf_{0 \leqq t < \infty} w(t) = 0. \tag{27}$$

This contradicts (20) and establishes the result. Q.E.D.

(b) In this part we eliminate condition (12). Let $w(t)$ be any positive function which is integrable in each finite segment. Define

$$w_\varepsilon(t) = \frac{1}{2\varepsilon} \int_{-\varepsilon}^{\varepsilon} w(t + \theta)d\theta. \tag{28}$$

where w_ε is a continuous positive function. Let

$$l_\varepsilon(t) = \int_{-L}^{L} w_\varepsilon(t+\theta)dP(\theta) = \frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon} l(t+\theta)d\theta. \tag{29}$$

If (10) were false then there would exist $\delta > 0$ and $t_0 < +\infty$ such that for each $t \geqq t_0$, $l(t) \leqq (1-\delta)w(t)$. But then for each $t \geqq t_0 + \varepsilon$, we would have

$$l_\varepsilon(t) \leqq (1-\delta)w_\varepsilon(t) \tag{30}$$

which contradicts the result established in part (a).

(c) In this part we eliminate condition (13). We consider the case where $P(\theta)$ is not supported by a finite segment. We may assume that $P(\theta)$ is not supported by $(-\infty, L)$ where L is finite. Condition (8) then implies that for each positive integer n it is possible to find a negative number d_n such that

$$\int_{d_n}^{n} \theta dP_n(\theta) = 0 \tag{31}$$

where $P_n(\theta)$ is a positive measure dominated by $P(\theta)$ such that the measure of any segment $[a,b]$ where $d_n < a < b \leqq n$ is the same under P and P_n. Let

$$\int_{d_n}^{n} dP_n(\theta) = F_n.$$

It follows from condition (8) that

$$\lim_{n\to\infty} F_n = 1. \tag{32}$$

Hence

$$\limsup_{t\to+\infty} \frac{\int_{-\infty}^{\infty} w(t+\theta)dP(\theta)}{w(t)} \geqq \limsup_{t\to+\infty} \frac{\int_{d_n}^{n} w(t+\theta)dP_n(\theta)}{w(t)} \tag{33}$$

$$= F_n \lim_{t\to+\infty} \frac{\int_{d_n}^{n} w(t+\theta)\,\frac{dP_n(\theta)}{F_n}}{w(t)} \geqq F_n$$

for each n. Now, (10) can be obtained from (32) and (33). Q.E.D.

By a similar method we can prove:

LEMMA 1 A. *If $P(\theta)$ is the distribution function of a probability measure satisfying*

$$\int_{-\infty}^{\infty} \theta dP(\theta) \leqq 0 \qquad (\textit{respectively}, \int_{-\infty}^{\infty} \theta dP(\theta) \geqq 0) \tag{34}$$

and $w(t)$ and $l(t)$ satisfy the conditions of Lemma 1, then

$$(35) \qquad \limsup_{t \to -\infty} (l(t) - w(t)) \geqq 0 \qquad (\textit{respectively}, \limsup_{t \to +\infty} (l(t) - w(t) \geqq 0).$$

LEMMA 2. *If $P(\theta)$, $w(t)$ and $l(t)$ satisfy the conditions of Lemma* 1 *and*

$$(36) \qquad l(t) \leqq w(t) \quad \text{for all } -\infty < t < \infty,$$

then

(a) *If $P(\theta)$ is not arithmetic, then $w(t) = C$ a.s.*

(b) *If $P(\theta)$ is arithmetic with span λ, then $w(t)$ is periodic having period λ.*

PROOF. Define a sequence z_i, $i = 1, 2, \cdots$ of independent random variables each of them having the distribution P. Let us denote

$$(37) \qquad s_n = \sum_{1}^{n} z_i$$

and

$$(38) \qquad y_n = w(s_n).$$

Condition (36) implies that $\{y_n\}$ is a positive submartingale. Hence there exists a random variable y such that

$$(39) \qquad y_n \to y \text{ with probability 1.}$$

We distinguish betwee two cases:

(a) $P(\theta)$ is not arithmetic.

In this case (8) implies that the random walk defined by (37) visits every interval infinitely often, with probability 1. This together with (39) implies that if w is a continuous function, then it has to be a constant.

If w is not continuous, we define

$$(40) \qquad w_\varepsilon(t) = \frac{1}{2\varepsilon} \int_{-\varepsilon}^{\varepsilon} w(t+\theta) d\theta.$$

Thus defined, w_ε is a continuous function satisfying the conditions of the lemma and so must be a constant C_ε. It is easily verified that C_ε has the same value C, for each ε, so that for all real t

$$(41) \qquad \lim_{\varepsilon \to 0} w_\varepsilon(t) = C.$$

On the other hand

$$(42) \qquad \lim_{\varepsilon \to 0} w_\varepsilon(t) = w(t) \qquad \text{a.s.}$$

which implies that $w(t) = C$ a.s. This proves part (a) of Lemma 2.

(b) $P(\theta)$ is arithmetic with span λ.

In this case (8) implies that the random walk defined by (37) visits every point $j \cdot \lambda$ (where j is any integer) infinitely often, with probability 1. Hence, (39) implies that $w(j \cdot \lambda)$ has the same value for each integer j.

In the same manner we can define:

$y_n = w(a + s_n)$, where a is any real number and deduce that $w(a + j\lambda)$ has the same value for every integer j. Q.E.D.

It should be noted that a similar result appears as a corollary in [2, p. 382]. The main differences are that in [2], $w(t)$ has to be bounded and there is an equality sign in (36). Instead of these requirements we impose condition (8). We now show that Condition (8) is necessary for the validity of the result of Lemma 2.

LEMMA 3. *Let $P(\theta)$ $-\infty < \theta < \infty$, be a probability measure satisfying*

$$\int_{-\infty}^{\infty} \theta dP(\theta) \neq 0, \tag{43}$$

then there exists a non-constant, positive, continuous and bounded function $w(t)$ which satisfies:

$$\int_{-\infty}^{\infty} w(t+\theta) dP(\theta) \leqq w(t), \qquad \text{for all } -\infty < t < \infty. \tag{44}$$

PROOF. We may assume that

$$\int_{-\infty}^{\infty} \theta dP(\theta) < 0. \tag{45}$$

Define a sequence z_i, $i = 1, 2, \cdots$ of independent random variables each of them having the distribution P. Let us denote

$$s_n = \sum_{1}^{n} z_i. \tag{46}$$

Define

$$w(t) = P(s_n \geqq -t \text{ for some } n \geqq 1), \quad -\infty < t < \infty. \tag{47}$$

It follows from the strong Law of Large numbers and (45) that

$$\lim_{t \to -\infty} w(t) = 0 \tag{48}$$

while obviously

$$\lim_{t \to \infty} w(t) = 1. \tag{49}$$

Thus, $w(t)$ is non-constant. Let us denote

$$l(t) = P(s_n \geqq -t \text{ for some } n \geqq 2) \tag{50}$$

then obviously

$$l(t) \leqq w(t). \tag{51}$$

On the other hand

$$\begin{aligned} l(t) &= \int_{-\infty}^{\infty} P_r(s_n \geqq -t-\theta \text{ for some } n \geqq 1)dP(\theta) \\ &= \int_{-\infty}^{\infty} w(t+\theta)dP(\theta). \end{aligned} \tag{52}$$

Hence $w(t)$ satisfies (44).

If $w(t)$ is continuous then it is the desired function. If not, define

$$w_\varepsilon(t) = \frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon} w(t+y)dy, \qquad \text{for some } \varepsilon > 0. \tag{53}$$

$w_\varepsilon(t)$ is a continuous function. We shall show that it satisfies (44):

$$\begin{aligned} w(t) &= \frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon} w(t+y)dy \\ &\geqq \frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon}\int_{-\infty}^{\infty} w(t+y+\theta)dP(\theta)dy \quad \text{(by (51) and (52))} \\ &= \int_{-\infty}^{\infty}\frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon} w(t+y+\theta)dydP(\theta) \quad \text{(because } w(t) \geqq 0) \\ &= \int_{-\infty}^{\infty} w_\varepsilon(t+\theta)dP(\theta) \end{aligned}$$

Q.E.D.

3. Proofs of the theorems

We now present the proofs of the Theorems presented in the introduction.

PROOF OF THEOREM 1. Let us denote the restriction of $A(\theta)$ to the segment $[-L, L]$ by A_L. We may assume that for each positive (finite) number L

$$\int_{-L}^{L} dA(\theta) < \infty. \tag{54}$$

Instead, if for a certain L, $\int_{-L}^{L} dA(\theta) = \infty$ then the positivity of $r(t)$ implies $\limsup_{t\to\pm\infty} (s(t))/(r(t)) = \infty$, so that (4) and (4′) hold.

Define

$$
(55)\qquad \begin{cases} g_L(b) = \displaystyle\int_{-L}^{L} e^{b\theta} dA_L(\theta) \\ q_L = \inf_{-\infty \leqq b \leqq \infty} g_L(b) \\ h_L = \limsup_{t\to-\infty} \dfrac{\displaystyle\int_{-L}^{L} r(t+\theta) dA_L(\theta)}{r(t)} \end{cases}
$$

and

$$
(56)\qquad \begin{cases} g(b) = \displaystyle\int_{-\infty}^{\infty} e^{b\theta} dA(\theta) \\ q = \inf_{-\infty < b < \infty} g(b) \\ h = \limsup_{t\to-\infty} \dfrac{s(t)}{r(t)} \end{cases}
$$

First we show that for each positive L

$$
(57)\qquad h_L \geqq q_L .
$$

We note that (54) implies that $g_L(b)$ is finite for each $-\infty < b < \infty$. If A_L is supported by the segment $[0, L]$ or by $[-L, 0]$, then $q_L = 0$ and (57) obviously holds. Hence we may assume that A_L is not supported by any of these intervals. In this case, $g_L(-\infty) = \infty$ and $g_L(+\infty) = \infty$. $g_L(b)$ is a differentiable positive function. If we denote its minimum point by $\bar{b}_L$, then

$$
(58)\qquad \int_{-L}^{L} \theta e^{\bar{b}_L\theta} dA_L(\theta) = g_L'(\bar{b}_L) = 0.
$$

Define a probability measure $P_L(\theta)$ by

$$
(59)\qquad dP_L(\theta) = \frac{e^{\bar{b}_L\theta}}{q_L} dA_L(\theta).
$$

It follows from (58) that

$$
(60)\qquad \int_{-L}^{L} \theta dP_L(\theta) = 0.
$$

If $w(t)$ is defined by $w(t) = r(t)e^{-\bar{b}_L t}$, then $w(t)$ satisfies the conditions of Lemma 1 and this together with (59) and (60) imply that

$$\limsup_{t\to-\infty} \frac{\int_{-L}^{L} r(t+\theta)dA_L(\theta)}{r(t)} \frac{1}{q_L}$$

$$= \limsup_{t\to-\infty} \frac{\int_{-L}^{L} w(t+\theta)e^{\bar{b}_L(t+\theta)}e^{-\bar{b}_L\theta}dP_L(\theta)}{w(t)e^{\bar{b}_L t}}$$

$$= \limsup_{t\to-\infty} \frac{\int_{-L}^{L} w(t+\theta)dP_L(\theta)}{w(t)} \geqq 1.$$

This proves (57).

We shall now prove that

$$h \geqq q. \tag{61}$$

where h and q are defined by (56). Assume that $q > 0$, then

(62) the measure A is not supported by a half line ($\theta \geqq 0$ or $\theta \leqq 0$).

Taking $L = 1, 2, \cdots n, \cdots$ in (58), we distinguish between two cases:

(a) $\bar{b}_n \to +\infty$ (or $\bar{b}_n \to -\infty$).

In this case it follows from (62) that $g_n(\bar{b}_n) \to +\infty$ and (57) implies that $h_L \to +\infty$ so that $h = \infty$.

(b) $\bar{b}_n \not\to +\infty$ (or $-\infty$)

In this case we can choose a subsequence $n_k \uparrow \infty$ such that

$$\bar{b}_{n_k} \underset{k\to\infty}{\to} \bar{b}_\infty \neq \pm\infty. \tag{63}$$

It follows from (55) and (56) that for any b,

$$g_L(b) \uparrow g(b) \text{ as } L \to \infty. \tag{64}$$

This inequality together with (57) implies that for all $k \geqq m$,

$$h > h_{n_k} \geqq g_{n_k}(\bar{b}_{n_k}) \geqq g_{n_m}(\bar{b}_{n_k}). \tag{65}$$

Since $g_{n_m}(b)$ is continuous, it follows from (65) that for each n_m, $h \geqq g_{n_m}(\bar{b}_\infty)$. This together with (64) implies that $h \geqq g(\bar{b}_\infty) \geqq q$. Q.E.D.

Proof of Theorem 2. Define a probability measure $P(\theta)$ and a positive function $w(t)$ by: $dP(\theta) = (e^{\bar{b}\theta}/g(\bar{b}))dA(\theta)$ and $w(t) = r(t)e^{-\bar{b}t}$.

Since $w(t)$ and $P(\theta)$ satisfy the conditions of Lemma 2, it follows that $w(t) = C(t)$ where $c(t)$ is a.s. a constant in case (a) and a periodic function in case (b).

Q.E.D

Note that if $g(b)$ is finite in an interval that includes $\bar{b}$ then condition (6) is satisfied. Lemma 3 implies that condition (6) which assures us that $r(t)$ is unique (up to multiplication by a positive function) is necessary. We shall return to this point in the examples of Chapter 5.

4. Discrete versions of the theorems

Both Theorem 1 and Theorem 2 hold for the discrete case. For brevity we state explicitly only the discrete version of Theorem 1.

THEOREM 1A. *Given two sequences* $\alpha_j \geqq 0$, $X_j > 0$, $-\infty < j < \infty$. *Then*

$$\limsup_{i \to -\infty} \frac{\sum_{j=-\infty}^{\infty} \alpha_j X_{i+j}}{X_i} \geqq \inf_{0 \leqq a \leqq \infty} \sum_{j=-\infty}^{\infty} \alpha_j a^j = q \tag{66}$$

and

$$\limsup_{i \to +\infty} \frac{\sum_{j=-\infty}^{\infty} \alpha_j X_{i+j}}{X_i} \geqq q. \tag{66'}$$

PROOF. Define a discrete measure $A(\theta)$ such that the measure of the point $\theta = j$ is α_j and the measure of any interval not containing integral points is zero. Define $r(t)$ to be the following step function: $r(t) = X_i$ for $i - 1 < t \leqq i$, $-\infty < i < \infty$. Thus defined, $A(\theta)$ and $r(t)$ satisfy the conditions of Theorem 1. Hence inequalities (66) and (66′) hold. Q.E.D.

It should be noted that the right side of (4) and (66) in Theorems 1 and 1A must include the end points ($b = \pm\infty$, $a = 0, \infty$). Otherwise the theorems would not be true when $q = \infty$. We show this by an example, which corresponds to Theorem 1A. Define $\alpha_j = 2^{j^2}$ for $j \leqq -1$ and $\alpha_j = 0$ for $j \geqq 0$. Then

$$\inf_{0 < a < \infty} \sum_{-\infty}^{-1} 2^{j^2} a^j = \infty.$$

On the other hand, for any $\varepsilon > 0$ we can find a sequence $\{X_i\}$ so that

$$\sup_{-\infty < i < \infty} \frac{\sum_{-\infty}^{-1} 2^{j^2} X_{i+j}}{X_i} < \varepsilon. \tag{67}$$

For example we can choose $X_i = (2N)^{-i^2}$ for $i \leqq -1$ where N is a number to be determined later. If $i < 0$, then

$$\frac{\sum_{-\infty}^{-1} 2^{j^2}(2N)^{-(i+j)^2}}{(2N)^{-i^2}} \leqq \frac{\sum_{-\infty}^{-1} 2^{j^2}(2N)^{-i^2-j^2}}{(2N)^{-i^2}}$$

$$\leqq \sum_{-\infty}^{-1} N^{-j^2} < \varepsilon$$

for sufficiently large N. It remains to note that for $i \geqq 0$, we can define X_i recursively so that (67) will hold for all i.

5. Examples

(a) Let $A(\theta)$ be an absolutely continuous measure defined as follows:

$$dA(\theta) = \begin{cases} \frac{2d}{\theta^3} d\theta & \text{for } \theta \geqq 1 \\ 0 & \text{for } 1 > \theta > -1 \\ \frac{2d-2}{\theta^3} d\theta & \text{for } \theta \leqq -1 \end{cases}$$

where $0 < d < 1$. Let $r(t)$ be a positive measurable function. The Bilateral Laplace Transform of $A(\theta)$, $g(b)$ defined by (5) is finite only for $b = 0$ so that $\bar{b}$ as defined by Theorem 2 is equal to 0 and $g(\bar{b}) = 1$. Assume that condition (7) is satisfied, i.e.,

$$\sup_{-\infty < t < \infty} \frac{\int_{-\infty}^{\infty} r(t+\theta)dA(\theta)}{r(t)} \leqq 1. \tag{68}$$

If $d = 1/2$, then condition (6) is satisfied and Theorem 2 then implies that $r(t) = C$ a.s.

If $d \neq 1/2$, then Lemma 3 implies that there exists a non-constant, continuous and bounded function $r(t)$, satisfying (68).

(b) Let $A(\theta)$ be defined as follows:

The measure of the point $\theta = 1$ is d where $d > 0$ and if $\theta \neq 1$ then

$$dA(\theta) = \begin{cases} -\frac{1}{\theta^3} d\theta & \text{for } \theta \leqq -1 \\ 0 & \text{for } -1 < \theta < 1 \\ & \text{and for } \theta > 1. \end{cases}$$

Let $r(t)$ be a positive function and assume that (7) holds. We note that $g(b)$ is finite for $b \geqq 0$ and satisfies

$$g(b) = d\,e^{b} - \int_{-\infty}^{-1} \frac{e^{b\theta}}{\theta^3}\,d\theta \tag{69}$$

$$\frac{dg(b)}{db} = d\,e^{b} - \int_{-\infty}^{-1} \frac{e^{b\theta}}{\theta^2}\,d\theta.$$

Since g is a concave function it follows that if

$$\frac{dg(b)}{db} \geqq 0 \qquad \text{at } b = 0 \tag{70}$$

then $\bar{b} = 0$ and if

$$\frac{dg(b)}{db} < 0 \qquad \text{at } b = 0 \tag{71}$$

then $0 < \bar{b} < \infty$.

Inequality (70) is equivalent to $d \geqq 1$. Thus if $d \leqq 1$, then (6) holds and $r(t) = C \cdot e^{\bar{b}t}$ a.s.

On the other hand if $d > 1$, then $\bar{b} = 0$ and Lemma 3 implies that there are other functions besides the constant functions which satisfy (7).

(c) Let $X_i, -\infty < i < \infty$, be a positive sequence. It follows from Theorem 1A that

$$\limsup_{i \to \pm\infty} \frac{X_{i-1} + X_{i+1}}{X_i} \geqq \inf_{0 \leqq a \leqq \infty} \left(\frac{1}{a} + a\right) = 2. \tag{72}$$

The discrete version of Theorem 2 implies that if

$$\sup_{-\infty < i < \infty} \frac{X_{i-1} + X_{i+1}}{X_i} = 2.$$

then $X_i = C$. This is a well known result. If a positive sequence X_i, $-\infty < i < \infty$, is concave, then $X_i = C$.

On the other hand, it is obvious that the condition:

$$\sup_{i > i_0} \frac{X_{i-1} + X_{i+1}}{X_i} = 2 \text{ where } i_0 \text{ is any fixed integer}$$

does not imply that the sequence is constant for $i > i_0$.

(d) Let X_i $-\infty < i < \infty$ be a positive sequence. It follows from Theorem 1A that

$$\limsup_{i\to\pm\infty} \frac{\sum_{-\infty}^{i+1} X_j}{X_i} \geqq \inf_{0<a\leqq\infty} \sum_{j=-\infty}^{1} a^j = 4. \tag{73}$$

The discrete version of Theorem 2 implies that if

$$\sup_{-\infty<i<\infty} \frac{\sum_{j=-\infty}^{i+1} X_j}{X_i} = 4$$

then $X_i = C.\,2^i$.

This particular result was established in [1] in connection with the optimal pure strategy for the Linear Search Game.

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